

# Calculation of the Aharonov-Bohm wave function <sup>\*</sup>

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## ABSTRACT

A calculation of the Aharonov-Bohm wave function is presented. The result is a series of confluent hypergeometric functions which is finite at the forward direction.

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# 1 Introduction

The scattering of charged particles by an infinitely long straight solenoid that encloses a magnetic flux, known as the Aharonov-Bohm effect, is of paramount interest in quantum physics [1]. Even though the region containing the magnetic field is inaccessible to the particles, the magnetic flux inside the solenoid affects their propagation. The observed interference pattern cannot be explained within classical physics; it is a purely quantal effect, without classical correspondence. This fact singularizes the Aharonov-Bohm (AB) effect from other important processes, like Rutherford scattering.

The AB effect has been analyzed by many different approaches. We shall restrict ourselves to the simplest situation of a straight filiform solenoid with constant magnetic flux. The object of interest is the wave function of the scattered particles and the ensuing scattering amplitude. There is some freedom in choosing the initial wave function, but we shall consider only two possibilities: a plane wave or a well-localized wave packet. In the first case we can drop the time dependence of the wave functions without losing any information; this situation will be referred to as “time-independent scattering”. In the second case we shall speak of “time-dependent scattering”. Most of the early analysis of the AB scattering can be considered time-independent.

The scattering amplitude was first calculated by Aharonov and Bohm [1] by means of a decomposition of the wave function in positive, negative and zero angular momentum components, which are calculated separately. The long-distance form of the scattered wave far from the forward direction defines the scattering amplitude that bears these authors’ names. They did not determine the wave function along the forward direction, where the scattering amplitude seems to diverge.

A different procedure is due to Berry [2], who obtained an exact single-valued wave function by summing the wave function endowed with the Dirac phase factor over all contributions (“whirling waves”) representing different windings round the flux line. It is also possible to obtain the same result by including the geometrical phase factor (Berry’s phase) instead of summing over whirling waves [3]. In these works, which have provided a thorough understanding of the AB effect, the wave function is presented as a series of Bessel functions which is not further elaborated.

A more explicit calculation of the AB wave function was supplied by Takabayashi [4]. An integral representation for the Bessel functions enabled this author to evaluate the asymptotic limit of the wave function far from the forward direction, thus reproducing the AB scattering amplitude, and the leading term of the wave function near the forward direction. Both the scattered and the transmitted wave were shown to be discontinuous along the forward direction, in such a way that the total wave function was continuous.

In the context of a more general investigation, Jackiw [5] found an integral representation for the total AB wave function similar to the one found in 2+1 dimensional gravity [6]. His integral representation allows for a natural decomposition of the total wave function in “transmitted” and “scattered” components. The transmitted wave was evaluated explicitly, but the scattered wave was calculated only in the asymptotic limit. The results agreed with those of [4].

Another derivation of the AB scattering amplitude and of the wave function near the forward direction has been provided by Dasnières de Veigy and Ouvry [7], this time by analyzing the action of the propagator on an incident plane wave. The procedure resembles those of [4] and [5], and yields identical results.

One of the few genuine time-dependent analyses of the AB effect has been given by Stelitano in a recent work [8]. This author decomposes the propagator in positive, negative and zero angular momentum components, and calculates each term independently in the asymptotic limit, following a procedure similar to the original method of Aharonov and Bohm [1]. His initial wave function

n is a well-localized Gaussian wave packet that approaches the solenoid. The final wave function is calculated as the convolution of the propagator with the initial wave packet. The resulting scattering amplitude is the same as in the time-independent analyses. The wave packet undergoes a self-interference along the forward direction analogous to the one found in [4] and [7].

It is clear that there is wide agreement about the results of these calculations. What we present in this work is a complete evaluation of the AB wave function based in the results of [5]. Our method is also applied to calculate the AB propagator. Therefore, both time-dependent and time-independent problems can be treated in a similar way. We shall produce analytic results in terms of confluent hypergeometric functions which are exact in any angular region, thereby being valid at the forward direction. This method was put forward recently in [9], where it was used to calculate the wave function in (2+1)-dimensional quantum scattering by a fixed spinless source. It is based on Pauli's article on the diffraction of light by a wedge limited by two perfectly reflecting planes [10].

This article is organized as follows. In Section 2 we shall review Jackiw's solution of the Schrödinger equation of a charged particle in presence of a magnetic vortex [5]. In a time-independent analysis the scattered wave function is given as an integral. The time-dependent propagator will be shown to be proportional to the same integral, which is therefore the object to calculate. That calculation is carried out in Section 3. The result is applied to the determination of the time-independent wave function in Section 4. The time-dependent analysis is presented in Section 5. The last Section contains a discussion of the results and our conclusions.

## 2 The wave function

In this Section we shall review briefly the solution given in [5] to the Schrödinger equation for the time-independent AB scattering problem. If we consider that the magnetic vortex coincides with the  $z$  axis, and that the incoming charged particles approach perpendicularly the vortex, the scattering process is essentially two-dimensional. In this situation a possible choice of vector potential is

$$\mathbf{A}(\mathbf{r}) = \frac{\Phi}{2\pi} \nabla \theta = \frac{c\hbar}{e} \nu \nabla \theta, \quad (1)$$

where  $\Phi$  is the flux carried by the vortex,  $\nu$  stands for the “numerical flux” defined as  $\nu = e\Phi/2\pi\hbar c$  and  $\theta$  is the polar angle of cylindrical coordinates. The Hamiltonian that defines the dynamics of the system is

$$\mathcal{H} = \frac{1}{2M} \left( \mathbf{p} - \frac{e}{c} \mathbf{A}(\mathbf{r}) \right)^2 \quad (2)$$

with  $\mathbf{A}(\mathbf{r})$  given by Eq. (1). The quantum mechanical evolution of the particle-vortex system is determined by the time-dependent Schrödinger equation,

$$i\hbar \frac{\partial}{\partial t} \Psi(r, \theta; \nu; t) = \mathcal{H} \Psi(r, \theta; \nu; t). \quad (3)$$

In a time-independent scattering process the Schrödinger equation (3) reduces to an energy eigenvalue problem if time is factorized in the usual way,

$$\begin{aligned} \Psi(r, \theta; \nu; t) &= e^{-itE/\hbar} \Psi_E(r, \theta; \nu) \\ \mathcal{H} \Psi_E(r, \theta; \nu) &= E \Psi_E(r, \theta; \nu). \end{aligned} \quad (4)$$

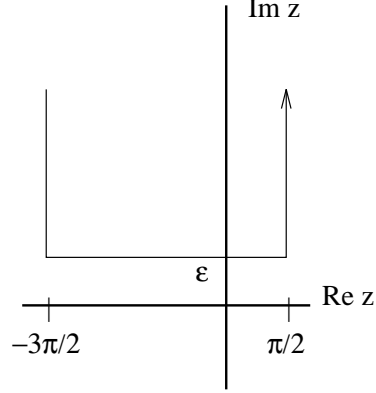


Figure 1: The Schlöfli contour

Since the spectrum of the Hamiltonian  $\mathcal{H}$  is continuous, with all energies  $E > 0$ , the interest resides in the eigenfunctions. These eigenfunctions are found to be [5]

$$\Psi_E(r, \theta; \nu) = \sqrt{\frac{M}{2\pi\hbar^2}} \left\{ \sum_{j=-\infty}^{[\nu]} e^{-i\pi(\nu+j)/2} J_{\nu-j}(kr) e^{ij\theta} + \sum_{j=[\nu]+1}^{\infty} e^{i\pi(\nu+j)/2} J_{j-\nu}(kr) e^{ij\theta} \right\}. \quad (5)$$

where  $k^2 = 2ME/\hbar^2$ . The brackets  $[\ ]$  indicate integer part. The sums can be performed with the help of the Schlöfli representation for the Bessel functions, whose contour of integration  $C_s$  is depicted in Fig. 1 ( $\epsilon$  is a small positive value for  $\text{Im}z$ ):

$$J_\alpha(x) = e^{i\alpha\pi/2} \int_{C_s} \frac{dz}{2\pi} e^{-ix \cos z} e^{iz\alpha}. \quad (6)$$

The sums in Eq. (5) are now geometric and can be evaluated in the usual way. After this summations and some changes of variable (we refer the reader to [5] for details), the wave function can be represented by

$$\Psi_E(r, \theta; \nu) = \sqrt{\frac{M}{2\pi\hbar^2}} e^{i\nu(\theta-\pi)} \int_C \frac{dz}{2\pi} e^{ikr \cos(z-\theta)} \frac{e^{-i\{\nu\}z}}{1 - e^{-iz}}. \quad (7)$$

The new contour  $C$  is shown in Fig. 2. It is easy to verify that the wave function (7) is single-valued, *i.e.* periodic in  $\theta$  with  $2\pi$  period. The contour avoids the poles  $z = 0 \bmod (2\pi)$ . An equivalent contour is depicted in Fig. 3. This new contour consists of a loop that encloses the pole  $z = 0$ , and two vertical lines. This separation corresponds to a decomposition of the total wave function  $\Psi_E(r, \theta; \nu)$  in two components, identified as the transmitted wave (closed contour) and the scattered wave (straight lines),

$$\begin{aligned} \Psi_E(r, \theta; \nu) &= \sqrt{\frac{M}{2\pi\hbar^2}} (\Psi_{\text{tr}}(r, \theta; \nu) + \Psi_{\text{sc}}(r, \theta; \nu)) \\ \Psi_{\text{tr}}(r, \theta; \nu) &= e^{ikr \cos \theta} e^{i\nu(\theta-\pi)} \\ \Psi_{\text{sc}}(r, \theta; \nu) &= e^{i[\nu]\theta} \sin(\pi\nu) \int_{-\infty}^{\infty} \frac{dy}{\pi} e^{ikr \cosh y} \frac{e^{\{\nu\}y}}{e^{y-i\theta} - 1}. \end{aligned} \quad (8)$$

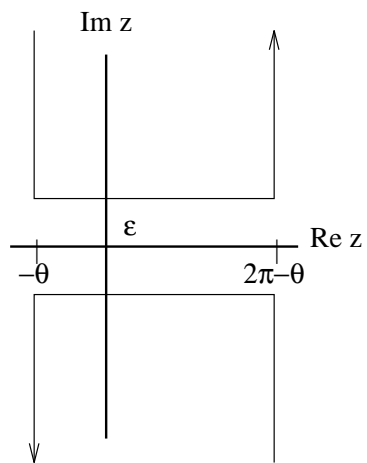


Figure 2: Contour of integration

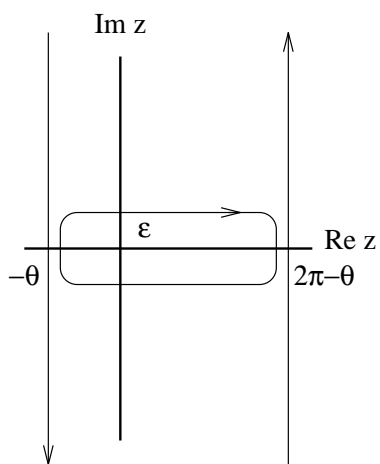


Figure 3: An equivalent contour

The transmitted wave has been evaluated by Cauchy's residue theorem. The scattered wave is an integral that can be calculated in the large  $kr$  limit by means of a saddle point approximation *except if*  $\theta = 0$ . If  $\theta = 0$  the integrand develops a singularity exactly at the saddle point  $y = 0$ , thus invalidating the procedure.

Before proceeding to the evaluation of the integral, we demonstrate that it also appears in the AB propagator. It has been shown in [8] that this propagator can be written as a Bessel series:

$$G(r, \theta; r', \theta'; \nu; t) = \frac{m}{2\pi i \hbar t} \exp \left\{ i \frac{m}{2\hbar t} (r^2 + r'^2) \right\} \sum_{n=-\infty}^{\infty} e^{in\phi} e^{-i|\nu-n|\pi/2} J_{|\nu-n|} \left( \frac{mrr'}{\hbar t} \right), \quad (9)$$

where  $\phi = \theta - \theta'$ . The sum can be transformed into an integral by means of the Schlöfli representation for the Bessel functions, Eq. (6). Following the same steps as for the time-independent wave function, the AB propagator can be decomposed in two terms that correspond to the transmitted and the scattered waves:

$$\begin{aligned} G_{\text{tr}}(r, \theta; r', \theta'; \nu; t) &= \frac{m}{2\pi i \hbar t} \exp \left\{ i \frac{m}{2\hbar t} |\mathbf{r} - \mathbf{r}'|^2 \right\} e^{i\nu\phi} \\ G_{\text{sc}}(r, \theta; r', \theta'; \nu; t) &= -\frac{m}{2\pi i \hbar t} \exp \left\{ i \frac{m}{2\hbar t} (r^2 + r'^2) \right\} e^{i[\nu]\phi} \sin(\{\nu\}\pi) \pi^{-1} \\ &\quad \times \int_{-\infty}^{\infty} dy \exp \left\{ i \frac{mrr'}{\hbar t} \cosh y \right\} \frac{e^{y\{\nu\}}}{1 + e^{y-i\phi}}. \end{aligned} \quad (10)$$

The computational difficulties lie now in the region  $\phi \approx \pm\pi$ , where a straightforward saddle-point approximation is ill-defined. The conclusion of this Section is that the evaluation of both the time-independent wave function and the time-dependent propagator reduces to a careful calculation of the same integral, which we shall denote by  $I(\rho, \theta, \{\nu\})$ ,

$$I(\rho, \theta, \{\nu\}) = \int_{-\infty}^{\infty} dy e^{i\rho \cosh y} \frac{e^{\{\nu\}y}}{e^{y-i\theta} - 1}. \quad (11)$$

### 3 Calculation of an integral

In this Section we shall calculate the integral  $I(\rho, \theta, \{\nu\})$  defined in Eq. (11). We are interested in an evaluation of this integral that be valid even if the angular variable  $\theta \in [0, 2\pi)$  takes values close to 0 or  $2\pi$ .

Let us change to a new variable  $s = e^{-i\pi/4} \sqrt{2} \sinh \frac{y}{2}$ , which extracts a Gaussian factor  $\exp(-\rho s^2)$  in the integrand. The path of integration can be taken as the real axis, so that the function  $I(\rho, \theta, \{\nu\})$  be

$$\begin{aligned} I(\rho, \theta, \{\nu\}) &= e^{i\pi/4} \sqrt{2} e^{i\rho} \int_{-\infty}^{\infty} ds e^{-\rho s^2} \frac{1}{1 - \cos \theta + is^2} \\ &\quad \times \frac{\cos \eta - \cos \theta}{e^{i(\eta-\theta)} - 1} \frac{e^{i\{\nu\}\eta}}{\cos \frac{\eta}{2}}. \end{aligned} \quad (12)$$

We introduce now the following notation:

$$\begin{aligned} f(s, \theta) &= \frac{\cos \eta - \cos \theta}{e^{i(\eta-\theta)} - 1} \frac{e^{i\{\nu\}\eta}}{\cos \frac{\eta}{2}} \\ a(\theta) &= e^{-i\pi} (1 - \cos \theta). \end{aligned} \quad (13)$$

The choice of phase in the definition of  $a(\theta)$  is not trivial since this function will appear below under a square root. The function  $I(\rho, \theta, \{\nu\})$  is now

$$I(\rho, \theta, \{\nu\}) = e^{-i\pi/4} \sqrt{2} e^{i\rho} \int_{-\infty}^{\infty} ds e^{-\rho s^2} \frac{f(s, \theta)}{ia(\theta) + s^2}. \quad (14)$$

The obvious procedure would be to expand the whole integrand in Eq. (14) except the Gaussian exponential, in powers of  $s$  and evaluate the Gaussian integrals. The result would be ill-defined at  $\theta = 0$ . An alternative procedure is to expand  $f(s, \theta)$  only, according to the following formula:

$$f(s, \theta) = \sum_{m=0}^{\infty} e^{im\frac{\pi}{4}} A_m(\theta) s^m \quad (15)$$

This expansion is well-defined for all values of  $\theta$ . The odd powers of  $s$  do not contribute to the integration in Eq. (14). The functions  $A_m(\theta)$  for  $m = 0, 2$  are

$$\begin{aligned} A_0(\theta) &= \frac{1}{2} (e^{i\theta} - 1) \\ A_2(\theta) &= \frac{1}{8} (2\{\nu\} - 1) (3 + e^{i\theta} - 2\{\nu\} + 2\{\nu\}e^{i\theta}) \end{aligned} \quad (16)$$

Now we insert the expansion (15) in Eq. (14) and consider  $\rho \neq 0$ . After the change of variable  $s = \tau \rho^{-1/2}$ , the function  $I(\rho, \theta, \{\nu\})$  reads

$$\begin{aligned} I(\rho, \theta, \{\nu\}) &= e^{-i\pi/4} \sqrt{2} e^{i\rho} \sum_{m=0}^{\infty} i^m A_{2m}(\theta) \rho^{\frac{1}{2}-m} \\ &\quad \times \int_{-\infty}^{\infty} d\tau e^{-\tau^2} \frac{\tau^{2m}}{ia(\theta)\rho + \tau^2} \end{aligned} \quad (17)$$

This final integration can be performed in terms of confluent hypergeometric functions as follows:

$$F_2 \left( 1, -m + \frac{3}{2}, ix \right) = \Gamma \left( m - \frac{1}{2} \right)^{-1} \int_{-\infty}^{\infty} d\tau e^{-\tau^2} \frac{\tau^{2m}}{ix + \tau^2}. \quad (18)$$

Therefore the final result is

$$\begin{aligned} I(\rho, \theta, \{\nu\}) &= e^{-i\pi/4} \sqrt{2} e^{i\rho} \sum_{m=0}^{\infty} i^m A_{2m}(\theta) \rho^{\frac{1}{2}-m} \\ &\quad \times \Gamma \left( m - \frac{1}{2} \right) F_2 \left( 1, -m + \frac{3}{2}, ia(\theta)\rho \right) \end{aligned} \quad (19)$$

The behaviour of the confluent hypergeometric functions defined in Eq. (18) for large or small  $|x|$  is

$$\begin{aligned} F_2 \left( 1, -m + \frac{3}{2}, ix \right) &\approx \left( m - \frac{1}{2} \right) (ix)^{-1} \left[ 1 - \left( m + \frac{1}{2} \right) (ix)^{-1} + \dots \right], & |x| \gg 1 \\ F_2 \left( 1, -m + \frac{3}{2}, ix \right) &\approx \left[ 1 + \left( -m + \frac{3}{2} \right)^{-1} ix + \left( -m + \frac{3}{2} \right)^{-1} \left( -m + \frac{5}{2} \right)^{-1} (ix)^2 + \dots \right] \\ &\quad + \pi \Gamma \left( m - \frac{1}{2} \right)^{-1} e^{-i(m\pi/2 + \pi/4)} x^{m-\frac{1}{2}} e^{ix}, & |x| \approx 0. \end{aligned} \quad (20)$$

By means of these formulas we can evaluate  $I(\rho, \theta, \{\nu\})$  in three limiting cases which will be important in the next Section:

### 3.1 $|\rho a(\theta)| \rightarrow \infty$

The leading term in this limit is

$$I(\rho, \theta, \{\nu\}) = -\sqrt{\frac{\pi}{2\rho}} e^{-i\pi/4} e^{i\rho} \frac{e^{i\frac{\theta}{2}}}{\sin \frac{\theta}{2}} + \dots \quad (21)$$

### 3.2 $\rho > 0$ and $\theta = 0^+$ or $2\pi^-$

As a shorthand we shall refer to these two possibilities as  $\theta = 0^+$  and  $\theta = 0^-$ , keeping in mind that  $\theta$  is an angular variable that takes value in the interval  $[0, 2\pi)$ . The leading contributions to  $I(\rho, \theta, \{\nu\})$  are:

$$I(\rho, 0^\pm, \{\nu\}) = \pm i\pi e^{i\rho} + (2\{\nu\} - 1) \sqrt{\frac{\pi i}{2\rho}} e^{i\rho} + \dots \quad (22)$$

The value of the discontinuity  $I(\rho, 0^+, \{\nu\}) - I(\rho, 0^-, \{\nu\})$  could have been determined from (11) by means of a simple distributional calculation based on the identity

$$\frac{1}{1 - e^{-y} \mp i\epsilon} = \mathcal{P} \frac{1}{1 - e^{-y}} \pm \pi i \delta(y), \quad (23)$$

where  $\epsilon \rightarrow 0^+$  and  $\mathcal{P}$  denotes the principal value. The  $\mathcal{O}(\rho^{-1/2})$  term in the right side of Eq. (22) vanishes if  $\{\nu\} = 1/2$ . That this is exact follows from a direct calculation using Eq. (23):

$$I(\rho, 0^\pm, \frac{1}{2}) = \pm i\pi e^{i\rho} \quad (24)$$

### 3.3 $\rho = 0$ and $\{\nu\} \neq 0$

This last case can be calculated directly from the definition of  $I(\rho, \theta, \{\nu\})$  and the identity (23). The result is

$$I(0, \theta, \{\nu\}) = -\pi e^{i\{\nu\}\theta} e^{-i\pi\nu} \frac{1}{\sin(\pi\nu)}. \quad (25)$$

To conclude this Section we wish to remark that we have found a unified expression for  $I(\rho, \theta, \{\nu\})$  that embraces all particular cases as limiting values of the parameters  $\theta$  and  $\{\nu\}$ , with the only exception of  $\rho = 0$ . We shall show in the next Section how these results can be successfully applied to the calculation of the AB wave function.

## 4 Application to time-independent AB scattering

We recall the solution to the static AB scattering problem found in [5]:

$$\begin{aligned} \Psi_{\text{tr}}(r, \theta; \nu) &= e^{i\nu(\theta-\pi)} e^{ikr \cos \theta} \\ \Psi_{\text{sc}}(r, \theta; \nu) &= e^{i[\nu]\theta} \sin(\pi\nu) \pi^{-1} I(\rho, \theta, \{\nu\}), \end{aligned} \quad (26)$$

where  $I(\rho, \theta, \{\nu\})$  is the function evaluated in the preceding Section. Therefore we can proceed directly to calculate the scattered wave function  $\Psi_{\text{sc}}(r, \theta; \nu)$  in some cases of interest.



#### 4.1 $kr(1 - \cos \theta) \rightarrow \infty$

This is the large  $kr$  limit away from the classical scattering direction  $\theta = 0$ . Using Eq. (21) we find

$$\Psi_{\text{sc}}(r, \theta; \nu) = \sqrt{\frac{i}{2\pi kr}} \exp \left\{ i[\nu]\theta + i\frac{\theta + \pi}{2} \right\} \frac{\sin(\pi\nu)}{\sin \frac{\theta}{2}} e^{ikr} + \dots \quad (27)$$

The large- $r$  asymptote of  $\Psi_{\text{sc}}(r, \theta; \nu)$  defines the scattering amplitude  $f(k, \theta)$  through the formula

$$\Psi_{\text{sc}}(r, \theta; \nu) \xrightarrow{r \rightarrow \infty} \sqrt{\frac{i}{r}} f(k, \theta) e^{ikr}. \quad (28)$$

Comparing Eqs. (27) and (28) we extract the usual AB scattering amplitude:

$$f(k, \theta) = \frac{1}{\sqrt{2\pi k}} \exp \left\{ i \left( [\nu] + \frac{1}{2} \right) \theta + i\frac{\pi}{2} \right\} \frac{\sin(\pi\nu)}{\sin \frac{\theta}{2}} \quad (29)$$

Other definitions of the scattering amplitude are possible, depending on what is meant by “incoming” and “scattered” waves. For instance we may consider that the incoming wave function is a plane wave  $e^{ikr \cos \theta}$ , and define the scattered wave as

$$\tilde{\Psi}_{\text{sc}}(r, \theta; \nu) = \Psi_{\text{sc}}(r, \theta; \nu) + \Psi_{\text{tr}}(r, \theta; \nu) - e^{ikr \cos \theta}. \quad (30)$$

*i.e.* the total wave function given by Eq. (8), minus the incoming plane wave. As explained in [5], the large  $r$  behaviour of this new scattered wave defines a scattering amplitude that differs from (29) in a  $\delta(\theta)$ -function, thus reproducing Ruijsenaar’s result in Ref. [11].

#### 4.2 $\theta = 0^\pm$ and $r > 0$

We are going to determine the wave function at the forward direction with the help of Eq. (22). The scattered wave develops a finite discontinuity,

$$\Psi_{\text{sc}}(r, 0^\pm; \nu) = \pm i \sin(\pi\nu) e^{ikr} + (2\{\nu\} - 1) \sqrt{\frac{i}{2\pi kr}} \sin(\pi\nu) e^{ikr} + \dots \quad (31)$$

As for the transmitted wave, we find directly from Eq. (26) that it presents a matching discontinuity:

$$\Psi_{\text{tr}}(r, 0^+; \nu) - \Psi_{\text{tr}}(r, 2\pi^-; \nu) = -2i \sin(\pi\nu) e^{ikr}. \quad (32)$$

Hence the total wave function  $\Psi_{\text{tr}}(r, \theta; \nu) + \Psi_{\text{sc}}(r, \theta; \nu)$  is continuous at the forward direction as it should be. It is possible to show that not only the discontinuities in the scattered and transmitted wave functions cancel, but also that all the discontinuities in the derivatives of one of them cancel against the discontinuities in the derivatives of the other. The outcome is that the analytic expression for the AB wave function presented in Eqs. (26) and (19) is smooth even at the forward direction, and can be used to examine the behaviour of the wave function for any value of  $\theta$ .

It can be said that the decomposition of the total wave function in “scattered” and “transmitted” components does not make sense at the forward direction (or in general at the classical scattering directions, see [9] for the situation in 2+1 dimensional gravitational scattering). We have seen that if we allow for separately discontinuous scattered and transmitted components of the total wave function, that decomposition can be sustained and yields a perfectly regular expression for

the total wave function. Therefore we can calculate its value at  $\theta = 0$  by averaging between  $0^+$  and  $2\pi^-$ . The result is

$$\Psi(r, 0; \nu) = \cos(\pi\nu)e^{ikr} + (2\{\nu\} - 1)\sqrt{\frac{i}{2\pi kr}} \sin(\pi\nu)e^{ikr} + \dots \quad (33)$$

It is interesting to note that the total time-independent wave function at the forward direction vanishes exactly if  $\{\nu\} = 1/2$  as a consequence of Eq. (24). That value for the fractional part of the numerical flux causes maximal scattering outside  $\theta = 0$ , and a total extinction of the wave function at  $\theta = 0$ . This can be understood as a consequence of probability conservation.

### 4.3 $r = 0$ and $\{\nu\} \neq 0$

Here we shall check that the wave function vanishes at the location of the magnetic vortex. Going back to Eq. (26) and taking into account Eq. (25) we find that the scattered and transmitted wave functions cancel out:

$$\begin{aligned} \Psi_{\text{tr}}(0, \theta, \nu) &= e^{i\nu(\theta-\pi)} \\ \Psi_{\text{sc}}(0, \theta, \nu) &= -e^{i\nu(\theta-\pi)} \end{aligned} \quad (34)$$

## 5 Time-dependent scattering

A similar analysis can be performed for the time-dependent scattering problem. The incoming wave function will be now a well-localized Gaussian wave packet of width  $\xi$  that approaches the magnetic vortex from a long distance  $r_0 \gg \xi$  at  $t = 0$ , with momentum  $\mathbf{k}$ :

$$\Psi_0(r', \theta'; 0) = \frac{1}{\sqrt{2\pi\xi}} \exp \left\{ ikr' \cos \theta' - \frac{1}{4\xi^2}(r'^2 + r_0^2 + 2rr' \cos \theta') \right\}. \quad (35)$$

In what follows we shall consider that  $kr_0 \gg 1$  and  $k \gg r_0/\xi^2$ . The angle  $\theta'$  is strongly confined at  $\theta' \approx \pi$ , and similarly  $r' \approx r_0$ . These approximations are needed to render feasible the following calculations. Therefore we shall not be able to ascertain the behaviour of the outgoing wave packet but in an approximate sense.

Having said that, the outgoing wave packet will be calculated as the convolution of the initial wave function (35) with the AB propagator,

$$\Psi(r, \theta; \nu; t) = \int_0^\infty r' dr' \int_0^{2\pi} d\theta' G(r, \theta; r', \theta'; \nu; t) \Psi_0(r', \theta'; 0). \quad (36)$$

We are therefore interested in an explicit evaluation of the propagator found in Sect. 2. The “scattered” propagator  $G_{\text{sc}}$  can be written in terms of the function  $I(\rho, \theta, \{\nu\})$  as

$$G_{\text{sc}}(r, \theta; r', \theta'; \nu; t) = \frac{m}{2\pi i \hbar t} \exp \left\{ i \frac{m}{2\hbar t} (r^2 + r'^2) \right\} e^{i[\nu]\phi} \sin(\{\nu\}\pi) \pi^{-1} I \left( \frac{mrr'}{\hbar t}, \phi + \pi, \{\nu\} \right), \quad (37)$$

where  $\phi = \theta - \theta'$ . Following the same steps as in the previous Section, we can evaluate  $G_{\text{sc}}$  in the following situations:

### 5.1 $\phi \approx \pi^-$ or $-\pi^+$

This case will be referred to as the “forward direction”. What we mean by this needs some clarification. If we are considering the propagator alone, there is no doubt that  $\phi = \pm\pi$  does indeed correspond to the forward propagation of an incoming particle. However, since our initial wave packet has a finite size, the forward direction is not well defined for the outgoing wave packet. The difficulty can be traced back to the fact that  $\theta'$  in Eq. (36) is a variable of integration, and therefore the condition  $\phi = \pm\pi$  does not define a fixed value for the outgoing angle  $\theta$ . A more realistic description makes use of the two length scales introduced in the scattering process by the initial wave packet (35):  $\xi$  and  $r_0$ . Qualitatively, the relevant values of  $\theta'$  are

$$\theta' \in \left[ \pi - \arctan \frac{\xi}{r_0}, \pi + \arctan \frac{\xi}{r_0} \right] \quad (38)$$

and therefore the condition  $\phi = \pi^-$  or  $-\pi^+$  means that the outgoing angle  $\theta$  is contained in the “forward cone”

$$\theta \in \left[ -\arctan \frac{\xi}{r_0}, \arctan \frac{\xi}{r_0} \right]. \quad (39)$$

Aside from these considerations, the situation is very similar to the time-independent scattering at the forward direction, and similar remarks apply. The propagator  $G_{\text{sc}}$  splits in two parts,

$$\begin{aligned} G_{\text{sc}}(r, \theta; r', \theta'; \nu; t) &= G_1^\pm(r, \theta; r', \theta'; \nu; t) + G_2(r, \theta; r', \theta'; \nu; t) \\ G_1^\pm(r, \theta; r', \theta'; \nu; t) &= \pm \frac{m}{2\pi\hbar t} \exp \left\{ i \frac{m}{2\hbar t} (r + r')^2 \right\} \sin(\nu\pi) \\ G_2(r, \theta; r', \theta'; \nu; t) &= (2\{\nu\} - 1) \sqrt{\frac{m}{8\pi^3 i \hbar t r r'}} \exp \left\{ i \frac{m}{2\hbar t} (r + r')^2 \right\} \sin(\nu\pi) + \dots \end{aligned} \quad (40)$$

The double sign  $\pm$  in  $G_1$  is  $+$  if  $\phi \approx -\pi^+$  and is  $-$  if  $\phi \approx \pi^-$ . We see that there is a finite discontinuity in  $G_1$  analogous to the one found in the time-independent analysis. This discontinuity is cancelled out by a similar discontinuity in  $G_{\text{tr}}$ , which is the propagator associated to the transmitted wave:

$$G_{\text{tr}}^+(r, \theta; r', \theta'; \nu; t) - G_{\text{tr}}^-(r, \theta; r', \theta'; \nu; t) = -\frac{m}{\pi\hbar t} \exp \left\{ i \frac{m}{2\hbar t} (r + r')^2 \right\} \sin(\nu\pi), \quad (41)$$

where the superindices  $+$  or  $-$  have the same meaning as in Eq. (40). Thus the total propagator is continuous at the forward direction. Also, all discontinuities in the derivatives of  $G_{\text{sc}}$  at the forward direction cancel with those of  $G_{\text{tr}}$ . The calculation of the total wave function at  $\theta = 0$  can therefore be performed by averaging between  $\theta = 0^+$  and  $\theta = 2\pi^-$ , with the following result :

$$\Psi(r, 0; \nu; t) = \cos(\pi\nu) \Psi_{\text{free}}(r, t) + (2\{\nu\} - 1) \sqrt{\frac{i}{2\pi k r}} \sin(\pi\nu) \Psi_{\text{free}}^R(r, t) + \dots \quad (42)$$

The notation  $\Psi_{\text{free}}(r, t)$  stands for a free wave packet that propagates along the forward direction, not to be confused with  $\Psi_{\text{free}}^R(r, t)$ , which propagates radially,

$$\begin{aligned} \Psi_{\text{free}}(r, t) &= \int_0^\infty r' dr' \int_0^{2\pi} d\theta' \exp \left\{ -i k r' - \frac{1}{4\xi^2} (r' - r_0)^2 \right\} \\ &\quad \times \frac{m}{2\pi i \hbar t} \exp \left\{ i \frac{m}{2\hbar t} (r + r')^2 \right\} \end{aligned}$$

$$\begin{aligned}\Psi_{\text{free}}^R(r, t) &= \int_0^\infty dr' \exp \left\{ -ikr' - \frac{1}{4\xi^2} (r' - r_0)^2 \right\} \\ &\times \sqrt{\frac{m}{2\pi i \hbar t}} \exp \left\{ i \frac{m}{2\hbar t} (r + r')^2 \right\}.\end{aligned}\quad (43)$$

A comparison between Eqs. (33) and (42) shows that the behaviour of the time-dependent and time-independent wave functions is similar at the forward direction. The leading contribution is modulated by the same factor  $\cos(\pi\nu)$ ; also, the  $\mathcal{O}(r^{-1/2})$ -term are identical.

## 5.2 $\phi \neq \pm\pi \bmod(2\pi)$ and $kr \rightarrow \infty$

The outgoing angle  $\theta$  is now outside the forward cone. In this angular region the transmitted wave is subdominant to the scattered wave, so that it suffices to take  $G_{\text{sc}}$  as propagator. With the help of Eq. (21) we find that the leading contribution to  $G_{\text{sc}}$  is

$$G_{\text{sc}}(r, \theta; r', \theta'; \nu; t) = -\sqrt{\frac{m}{8\pi^3 i \hbar t r r'}} \exp \left\{ i \frac{m}{2\hbar t} (r + r')^2 \right\} e^{i[\nu]\phi} \sin(\{\nu\}\pi) \frac{e^{i\frac{\phi}{2}}}{\cos\frac{\phi}{2}} + \dots \quad (44)$$

An exact evaluation of Eq. (36) with  $G_{\text{sc}}$  and  $\Psi_0$  given by Eqs. (44) and (35) respectively is not known to us. In this circumstance we can resort to the approximation of taking  $\theta' = \pi$  in  $G_{\text{sc}}$ , but not in the wave function  $\Psi_0$ . Within this approximation the leading (large  $kr$ ) contribution to the scattered wave function can be easily calculated, with the following result:

$$\Psi_{\text{sc}}(r, \theta; \nu; t) = \sqrt{\frac{i}{2\pi kr}} \exp \left\{ i \left( [\nu] + \frac{1}{2} \right) \theta \right\} \frac{i \sin(\pi\nu)}{\sin\frac{\theta}{2}} \Psi_{\text{free}}^R(r, t), \quad (45)$$

where  $\Psi_{\text{free}}^R(r, t)$  is the freely propagating one-dimensional (radial) wave packet defined in Eq. (43).

Again, the AB scattering amplitude can be derived from Eq. (45). The result is of course Eq. (29).

## 6 Conclusions

The general solution [5] to the time-independent AB wave function, Eq. (8), has been evaluated as a series of confluent hypergeometric functions (19, 26). The same procedure can be applied to the AB propagator, wherefrom the propagation of a Gaussian wave packet has been determined. The method is inspired by a recent work [9] on (2+1)-dimensional gravitational scattering, which in its turn was based on Pauli's article [10] on diffraction of light. We have applied that procedure to determine the AB wave function far from and along the forward direction. Our results agree with those of previous analyses, both in the time-independent or time-dependent approach [4, 7, 8].

The factor  $\cos(\pi\nu)$  present in Eqs. (33) and (42) is usually interpreted as a self-interference between two “halves” of the incoming wave that are split by the magnetic vortex and recombined at the forward direction. Each half carries a phase  $\exp(\pm i\pi\nu)$ , whose addition produces the  $\cos(\pi\nu)$ . This explanation is only heuristic, since the scattered wave is not taken into account. We have seen that the peculiar behaviour of the scattered wave in the forward direction, Eq. (31), is crucial in rendering the total wave function continuous and single-valued at  $\theta = 0$ .

On the other hand, the separation of the wave function in “scattered” and a “transmitted” components becomes arbitrary precisely at the forward direction. The first term in Eq. (31) could

as well be considered part of the transmitted wave. Our opinion is that these descriptions of the A B effect rely on a semiclassical conception of the wave function, which is supposed to be separable in parts that propagate independently. The calculation presented here recovers the essential unity of the wave function after summing the scattered and transmitted components, and only then we find the distinctive behaviour of the AB wave function in the forward direction.

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### References

- [1] Y. Aharonov, D. Bohm, *Phys. Rev.* **115**, 485 (1959)
- [2] M. V. Berry, *Eur. J. Phys.* **1**, 240 (1980)
- [3] M. V. Berry, *Proc. R. Soc. Lond.* **A392**, 45 (1984)
- [4] T. Takabayashi, *Hadronic Journal Supplement* **1**, 219, (1985)
- [5] R. Jackiw, *Ann. Phys. (N. Y.)* **201**, 83 (1990)
- [6] S. Deser, R. Jackiw, *Comm. Math. Phys.* **118**, 495 (1988)
- [7] A. Dasnières de Veigy, S. Ouvry, *C. R. Acad. Sci. Paris* **t.318**, Série II, 19 (1994)
- [8] D. Stelitano, *Phys. Rev.* **D51**, 5876 (1995)
- [9] M. Alvarez, F. M. de Carvalho Filho, L. Griguolo *MIT-CTP #2451*, hep-th/9507134, To appear in *Comm. Math. Phys.*
- [10] W. Pauli, *Phys. Rev.* **54**, 924 (1938)
- [11] S. N. M. Ruijsenaars, *Ann. Phys. (N. Y.)* **146**, 1 (1983)